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Optimization of a Quasi-Zero-Stiffness Isolator

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Abstract

The frequency range over which a mount can isolate a mass from a vibrating base (or vice versa) is often limited by the mount stiffness required to support the weight of the mass. This compromise can be made more favourable by employing non-linear mounts with a softening spring characteristic such that small excursions about the static equilibrium position result in small dynamic spring forces and a correspondingly low natural frequency. This paper concerns the force-displacement characteristic of a so-called quasi-zero-stiffness (QZS) mechanism which is characterised by an appreciable static stiffness but very small (theoretically zero) dynamic stiffness.

The mechanism studied comprises a vertical spring acting in parallel with two further springs which, when inclined at an appropriate angle to the vertical, produce a cancelling negative stiffness effect. Analysis of the system shows that a QZS characteristic can be obtained if the system's parameters (angle of inclination and ratio of spring stiffnesses) are opportunely chosen. By introducing the additional criterion that the displacement of the system be largest without exceeding a desired (low) value of stiffness an optimal set of parameter values is derived. Under sufficiently large displacements the stiffness of the QZS mechanism can eventually exceed that of the simple mass-spring system and criteria for this detrimental scenario to arise are presented.

Keywords: Vibration isolation; Quasi-zero-stiffness

1. Introduction

Isolation of undesirable vibrations is a problem that affects many engineering structures. In the ideal case of a mass m supported by a linear stiffness k on a rigid foundation, isolation does not occur until a frequency of $\sqrt{2k/m}$. It is evident that a smaller stiffness results in a wider frequency range of isolation. However, a smaller stiffness results in a larger static displacement of the mass, and this trade-off between isolation and static displacement is well-known (Den Hartog, 1956). To overcome this limitation non-linear springs have been used to obtain a high static stiffness and hence a small static displacement, and a small dynamic stiffness, which results in a low natural frequency (Rivin, 2001; Alabuzhev et al., 1989). This is generally achieved by configuring springs so that they act as a negative stiffness in parallel with a positive stiffness (Platus, 1991; Zhang et al., 2004). By careful choice of geometry and stiffness it is possible to achieve an isolator with zero dynamic stiffness, a so-called quasi-zero-stiffness (QZS) mechanism (Alabuzhev et al., 1989). Applications of QZS mechanisms range from space research, e.g., to simulate zero gravity (Denoyer and Johnson, 2001), to isolation of high precision machinery (Dankowski, 2001).

The simplest non-linear mount that contains the essential elements is shown in Fig. 1 in its unloaded condition. When it is loaded with a suitably sized mass, the springs compress such that the oblique springs, k_o are in the horizontal position and the static load is taken by the vertical spring, k_v. This is the

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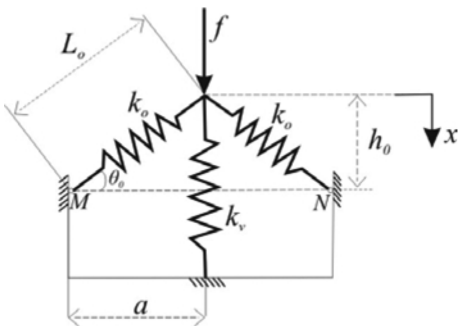

Fig. 1. Schematic representation of the simplest system with quasi-zero stiffness mechanism.

static equilibrium position, and it is the motion about this position that is of primary interest. When the system of springs is used in this way, the oblique springs act as a negative stiffness in the vertical direction counteracting the positive stiffness of the vertical spring.

The combination of these effects produces a quasi-zero-stiffness characteristic. In this paper an optimum relationship between the ratio of the oblique spring stiffness and the vertical spring stiffness is sought, as is an optimum angle for the oblique springs.

2. Force-displacement characteristic of a system with two oblique springs

It is instructive to examine first the behaviour of the oblique springs alone. Consider the system in Figure 1, but with the vertical spring k_v removed. The two linear springs each of stiffness k_o hinged at points *M* and *N* respectively have initial length L_0. A force *f* is applied at point *P* which is a horizontal distance *a* from points *M* and *N* and initially at height h_0 above these points. The springs are initially at an angle θ_0 from the horizontal. The vertical component of the applied force is related to the spring stiffness k_o by

$$f = 2k_o(L_0 - L)\sin\theta \quad (1)$$

where *L* is the length of the compressed spring, and $\sin\theta = (h_0 - x)/L$.

From the geometry of the system Eq. (1) can be written as

$$f = 2k_o(h_0 - x)\left(\frac{\sqrt{h_0^2 + a^2}}{\sqrt{(h_0 - x)^2 + a^2}} - 1\right) \quad (2)$$

which can be written in non-dimensional form as

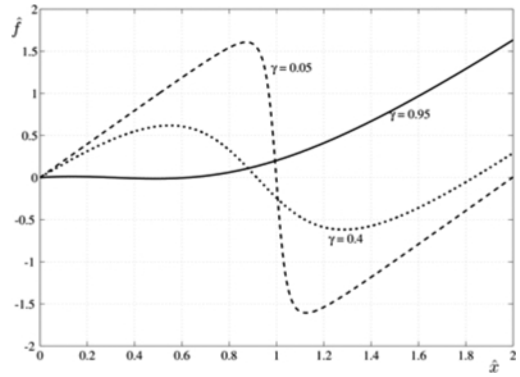

Fig. 2. Force-deflection characteristic of the system in Figure 1. When $\gamma = 0$ the springs are vertical and when $\gamma = 1$ they are horizontal. The stiffness is negative between the maxima and the minima.

$$\frac{f}{k_o L_0} = 2(\sqrt{1-\gamma^2} - \hat{x})\ \left\{\left[\hat{x}^2 - 2\sqrt{1-\gamma^2}\,\hat{x} + 1\right]^{-1/2} - 1\right\} \quad (3)$$

where $\hat{x} = x/L_0$ and

$$\gamma = \frac{a}{L_0} = \cos\theta_0 \quad (4)$$

is a *geometrical parameter*. When $\gamma = 0$ the springs are initially vertical and when $\gamma = 1$ the springs, initially, lie horizontally. Figure 2 shows the non-dimensional force plotted against the non-dimensional displacement for different values of γ.

It can be seen that the system has a highly non-linear characteristic. Between the peaks the stiffness of the system is negative. There is not a static equilibrium position in this region, and the mechanism will "snap through" to a stable position if it is forced into this region.

The system can be modified to exhibit QZS by adding a vertical spring of equal and opposite (positive) stiffness. Such a system is the focus of the following section.

3. A QZS mechanism

For the system in Fig. 1, the vertical spring k_v is in parallel with the vertical components of the oblique springs. Choosing now to non-dimensionalise force by $k_v L_0$, the non-dimensional spring force $\hat{f}$ is given by

$$\hat{f} = \hat{x} + 2\alpha(\sqrt{1-\gamma^2} - \hat{x})\ \left\{\left[\hat{x}^2 - 2\sqrt{1-\gamma^2}\,\hat{x} + 1\right]^{-1/2} - 1\right\} \quad (5)$$

where $\alpha = k_0/k_v$ is the ratio of the spring stiffnesses.

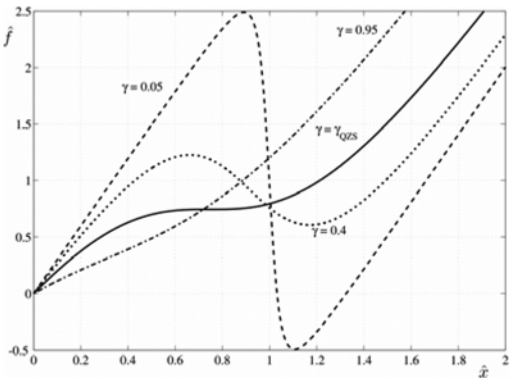

Fig. 3. Force-displacement characteristic of a QZS mechanism when $\alpha = 1$: the solid line is the QZS system.

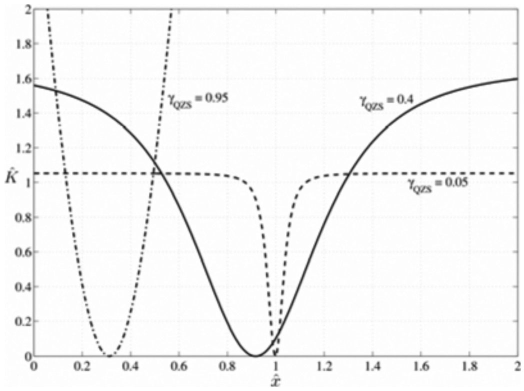

Fig. 4. Non-dimensional stiffness for different combinations of geometrical and stiffness parameters that yield QZS.

For large α Eq. (5) tends to Eq. (3). The non-dimensional force as a function of the non-dimensional displacement is plotted in Fig. 3 for several values of γ and for $\alpha = 1$.

It can be seen that for a given value of α, there is one particular value of the geometrical parameter, denoted as γ_{QZS}, for which the stiffness is always positive except at one position where it is zero.

The stiffness of the system can be found by differentiating Eq. (5) with respect to the displacement to give

$$\hat{K} = 1 + 2\alpha\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\,\hat{x} + 1\right)^{3/2}}\right]. \tag{6}$$

If Eq. (6) is evaluated at the static equilibrium position $\hat{x}_e = \sqrt{1-\gamma^2}$ and set to zero, then the value γ_{QZS} that gives zero-stiffness is

$$\gamma_{QZS} = \frac{2\alpha}{2\alpha + 1} \tag{7a}$$

for a given value of α. Equivalently, the value of α that ensures QZS behaviour for a given γ is

$$\alpha_{QZS} = \frac{\gamma}{2(1-\gamma)}. \tag{7b}$$

4. Optimisation of the QZS mechanism

By enforcing the QZS condition on α and γ in Eq. (7), the non-dimensional stiffness given by Eq. (6) can be written as a function of just the geometrical parameter as

$$\hat{K}_{QZS} = 1 + \frac{\gamma_{QZS}}{(1-\gamma_{QZS})}\left[1 - \frac{\gamma_{QZS}^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma_{QZS}^2}\,\hat{x} + 1\right)^{3/2}}\right] \tag{8}$$

This is plotted in Fig. 4 for several values of γ_{QZS}. It can be seen from the figure that the stiffness is zero at the static equilibrium position, $\hat{x}_e$, and the displacement range over which there is a small stiffness depends on γ_{QZS}.

Of interest is the range of displacements about the equilibrium position for which the stiffness is less than a prescribed stiffness $\hat{K}_o$, say. (Note that a value of $\hat{K}_o = 1$ means that the stiffness of the system is equal to that of the vertical spring).

$$\hat{x}\big|_{\hat{K}=\hat{K}_o} = \hat{x}_e \pm \hat{d} \tag{9}$$

where $\hat{x}_e$ is the static equilibrium position and d is the excursion, normalised by L_0, from this position when $K = K_o$, and is given by

$$\hat{d} = \gamma_{QZS}\sqrt{\left[\frac{1}{1-\hat{K}_o(1-\gamma_{QZS})}\right]^{2/3} - 1}. \tag{10}$$

In order to determine the maximum $\hat{d}$ that can be achieved for a given $\hat{K}_o$ as γ_{QZS} changes from 0 to 1 Eq.(10) has to be differentiated and set to zero. The complete analysis is reported in (Carrella *et al.,* 2006) and shows that there is only a weak relationship between the optimum geometry and the prescribed maximum stiffness of the system in which the angle for the oblique springs ranges from about 48° to 57°. The corresponding optimum stiffness ratio α_{opt} ranges from 1 to 0.6.

Although there are benefits to incorporating springs configured to act as a negative stiffness, there are also some disadvantages. The oblique springs only act as a negative stiffness over a certain displacement range. Outside this range they act as a positive stiffness, adding to the stiffness of the vertical spring. The peak positive stiffness can be obtained by setting $x \gg h_0$ such that $\hat{x} \gg 1$ and Eq. (8) becomes

$$\hat{K}\Big|_{x \gg h_0} \approx \frac{1}{1-\gamma_{QZS}} \tag{11}$$

The optimal value for γ_{QZS} lies between 2/3 and $(2/3)^{3/2}$ depending on the stringency with which low stiffness is required. Thus, the cost of having a QZS mechanism is that for large excursions from the static equilibrium position the stiffness can increase to between about two and three times that of the vertical spring.

5. Conclusions

The static characteristics of a quasi zero stiffness mechanism have been investigated. The main feature of such a mechanism is the use of a negative stiffness element to achieve a low stiffness without having a large static deflection. A simple system consisting of three springs has been studied, and the optimum relationship between the geometry and the relative stiffnesses of the springs has been investigated. It has been found that to achieve a large excursion from the static equilibrium position such that the stiffness of the system does not exceed a prescribed value, there is an optimum geometry and a corresponding optimum relationship between the stiffnesses. It has also been shown that the drawback is a stiffness higher than the vertical spring, should the displacement from the static equilibrium position become large.

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